Abstract

In this thesis, we analyze a coupled system of the form

$$\begin{cases} (\mathcal{E}u)'(t) + \phi_1(t, u(t), v(t)) = q_1(t), \\ v''(t) - \Delta v(t) + \phi_2(t, u(t), v(t)) = q_2(t). \end{cases} \tag{1.1}$$

It consists of a semilinear abstract differential-algebraic equation (DAE) and a semilinear second order hyperbolic partial differential equation. Both equations are coupled through the nonlinear coupling functions ϕ_1 and ϕ_2.

Coupled systems of the form (1.1) can be interpreted as a specific kind of abstract DAE or as a generalization to partial differential-algebraic equations. They are relevant for a variety of applications, for instance the modeling of multiphysics systems, the simulation of circuits, or the optimal control of gas flow through a pipe network.

In this thesis, we first discuss only the semilinear abstract DAE, and introduce so-called matrix-induced linear operators. Using these operators, we transfer a decoupling strategy developed for DAEs to the infinite-dimensional setting of abstract DAEs. In combination with a novel index-1 characterization for semilinear abstract DAEs, this allows to extract from the abstract DAE the inherent ordinary differential equation and the complementing algebraic equations. We then prove existence and uniqueness of solutions.

We combine the developed analytical techniques for semilinear abstract DAEs with matrix-induced linear operators with analytical tools known from the theory of second order hyperbolic equations to provide a framework suitable for the analysis of system (1.1). By means of a fixed-point approach, we show existence and uniqueness of local and global solutions.

Finally, we formulate an optimal control problem where system (1.1) acts as a side condition. We show the existence of an optimal control and a global minimizer.

1 Introduction

Many natural phenomena are modeled by differential equations, whether it is to describe the interaction between physical particles, or to understand how chemical substances react and diffuse, whether it is to model the spreading of a new virus, or to predict the climate change or local weather. The ever-increasing computational power and the availability of larger and larger sets of data allow to describe all of these phenomena using better suited and more complex sets of differential equations in which the describing components of the underlying physical, chemical, or biological system are intimately coupled. Understanding these systems and knowing how to influence them in a desirable way helps to develop new strategies, e. g. for how to practice agriculture in a more sustainable and cost efficient way, it helps to predict consequences of events like earthquakes, hurricanes, and inundation, and it indicates how to counteract for instance undesirable effects of long-term medication.

With this thesis we try to help towards a better understanding of coupled systems of different kinds of differential equations. More specifically, we analyze a coupled system of an abstract differential-algebraic equation (DAE) and a specific second-order hyperbolic partial differential equation (PDE), the wave equation. It takes the form

$$\begin{cases} (\mathcal{E}u)'(t) + \phi_1(t, u(t), v(t)) = q_1(t), & (1.1a) \\ v''(t) - \Delta v(t) + \phi_2(t, u(t), v(t)) = q_2(t). & (1.1b) \end{cases}$$

This coupled system consists of the semilinear abstract DAE (1.1a) and the semilinear wave equation (1.1b). The two solution variables u and v are both functions of the time t and of spatial variables not explicitly stated here. The linear operator $\mathcal{E}$ of (1.1a) is a so-called matrix-induced linear operator introduced in Chapter 2, and the coupling functions ϕ_1 and ϕ_2 are nonlinear but continuous. The system can be manipulated through right-hand side functions q_1 and q_2. For the analysis in this thesis, it will be complemented by appropriate initial and boundary conditions.

The main goal of this thesis is the analysis of the system (1.1). In particular, we want to provide a framework in which existence and uniqueness of local and global solutions can be ensured. The perhaps most challenging task in this analysis is to find a common setting in which all components of the coupled systems can be discussed satisfactorily. In the course of mathematical research history, the analytical techniques and tools developed to analyze a specific differential equation became more and more tailored and bespoke. Our intent is, in a sense, to go a step in the opposite direction, to see if it is possible to consolidate the different settings for abstract DAEs and hyperbolic second order PDEs, and to strive towards a more unified framework which is equally suited for both. Thus, we are driven not only by an external but also an inner-mathematical motivation.

Main Contributions

We want to emphasize the main contributions of this thesis. First, we develop the notion of so-called *matrix-induced linear operators*. Although these kinds of operators appear frequently but implicitly in the research literature on abstract DAEs, e. g. [86, 128], and although they promise to be very useful, particularly for the analysis of coupled systems, they have not been discussed in the context of abstract DAEs so far. Using this kind of operators, we are able to translate a decoupling approach that was developed for DAEs in [64] to the infinite-dimensional framework of abstract DAEs. In combination with a novel theoretical existence result for a certain type of operator equation, see Theorem 2.20, this decoupling approach allows to prove existence and uniqueness of strong solutions for a semilinear abstract DAE of the form (1.1a).

Second, we provide a framework for the coupled system (1.1). To this end, we first discuss a related system where the wave equation (1.1b) is coupled with an abstract ordinary differential equation (ODE) instead of (1.1a). We prove existence and uniqueness of local as well as global solutions to this related coupled system by means of a fixed-point approach. Afterwards, we use the techniques developed in Chapter 2 to transfer the results to system (1.1).

Third, we take a glance at an optimal control problem which is constrained by the related coupled system of abstract ODE and wave equation. We discuss whether the framework previously chosen for the analysis of (1.1) is equally appropriate for the optimal control problem, and we show under strong assumptions that the optimal control problem admits a global minimizer.

Structure and Literature

Observe that each chapter is more or less similarly structured. Due to the inherent consolidating character of this thesis, each chapter starts with a detailed introduction into the chapter's general topic. We then give an overview of the contributions of the chapter and integrate our results into existing research literature. Therefore, we will keep this overview short.

Chapter 2 is dedicated to our first main contribution, the analysis of a semilinear abstract DAE of the form (1.1a). We introduce the concept of matrix-induced linear operators, define appropriate solution spaces, and prove existence and uniqueness of a solution. The work of this chapter can be seen as a continuation and an addition to the research done by Tischendorf [119] and Matthes [86], but is also related to [9, 128].

In Chapter 3, we give an introduction into the topic of second order hyperbolic equations. We present certain general techniques for the analysis of such equations, we highlight characteristic features, and we apply these results to the special case of the prototypical linear wave equation. This serves as a justification to use Equation (1.1b) as a representative of a larger class of second order hyperbolic PDEs.

Chapter 4 is dedicated to our second main contribution. First, we provide a suitable framework for a related coupled system of abstract ODE and wave equation, and we provide existence and uniqueness results under specific assumptions on the coupling functions ϕ_1 and ϕ_2. Afterwards, we transfer the results obtained to coupled systems of the form (1.1).

Finally, in Chapter 5, we take a first step into an optimal control problem where the coupled system of abstract ODE and wave equation related to (1.1) serves as a restriction. We are able to show the existence of an optimal control and a global minimizer for a specific cost functional. We do not derive first or higher order conditions.

This thesis is complemented by three appendices. In Appendix A, we recall the intricate relation between certain matrix factorizations, generalized inverses of matrices, and projections onto and along certain subspaces. In Appendix B, we collect tools and knowledge from functional analysis, in particular from the theory of Bochner spaces. In Appendix C, we recall existence results for abstract differential equations and operator equations.

Citation and Notation

Our aim is to make this thesis as consistent as possible to provide for a pleasant lecture. This applies to citations as well, which is why most statements we took from literature are not cited verbatim. Nevertheless, we always indicate where a certain statement can be found.

Throughout this thesis, $[0,T] \subset \mathbb{R}$ always denotes a finite time interval with $T > 0$. The dimension of the spatial domain $\Omega \subset \mathbb{R}^d$ is consistently denoted with $d \in \mathbb{N}$. The solution variable for ODEs and DAEs is u; the solution variable for PDEs is usually v. If u is vector-valued, it maps either to $\mathbb{R}^n$ or $\mathbb{R}^r$. The specific meanings of the natural numbers $n \in \mathbb{N}$ and $r \in \mathbb{N}$ will become clear in Chapter 2.

A general Banach space is denoted by $(X, \|\cdot\|_X)$. Following [124], we denote the dual space of X with $(X', \|\cdot\|_{X'})$. We use $|\cdot|$ exclusively for the Euclidean norm in the finite-dimensional vector space $\mathbb{R}^n$. In all other cases, also for general Hilbert spaces, the norm is denoted by $\|\cdot\|$. We indicate the specific norm by a subscript; the only exceptions to this rule are matrix norms which have to do without. The transpose of a matrix A is denoted by A^{T}.

A general Hilbert space is denoted by $(H, (\cdot\,,\cdot))$. We explicitly distinguish between dual pairings $\langle\cdot\,,\cdot\rangle_X$ and inner products $(\cdot\,,\cdot)_H$. If unambiguous, we drop the subscript for dual pairings and inner products. This holds true also and in particular when we use Gelfand triples (X, H, X'), see Definition B.2.

If a Banach space X is embedded in another Banach space Y, we write $X \hookrightarrow Y$. In this thesis, embeddings are always topological embeddings, i. e. they are injective and continuous. If an embedding is dense or compact, we do not use a specific notation but rather write it out explicitly.

Given a function $v\colon [0,T] \to X$, we denote with v' its first derivative with respect to time. Since we do not identify certain Lebesgue-Bochner spaces with Lebesgue spaces, for instance, we always write $L^2(0,T;L^2(\Omega))$ and never $L^2((0,T)\times\Omega)$, the notation for the time derivative is unambiguous. Other partial derivatives or normal derivatives are written out explicitly. In this thesis, in particular when using Sobolev spaces or Bochner spaces of weakly differentiable abstract functions, we avoid the use of distributional derivatives. For our purposes, the notion of weak derivatives is sufficiently general.

Apart from the common abbreviations, we only use two more. In formulas, we write f. a. a. instead of "for almost all", and we use a. e. instead of "almost everywhere".

Finally, we would like to explain one specific notational decision. Throughout this thesis, we collect necessary assumptions separately. This permits to simply refer to these assumptions at the beginning of definitions, theorems, and so on. It also helps to keep the assertions concise, and it allows to base one assumption upon another. Therefore, we decided to number the assumptions consecutively without referring to the chapter in which the assumption first appeared. Unfortunately, this makes our assumptions harder to find, which is why we provided a list of assumptions directly subsequent to the table of contents.

We conclude the introduction by stating the first and most fundamental assumption which is supposed to hold throughout the entire thesis.

Assumption 1. Let $[0,T] \subset \mathbb{R}$ be a given fixed time interval with $T > 0$, and let $\Omega \subset \mathbb{R}^d$ be an open interval for $d = 1$, and a Lipschitz domain for $d \in \{2,3\}$, see Definition B.10.